4.4 (Poole)

Similarity and Diagonalization

Consider the matrices

$$A = \begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix}, \qquad D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 13 & 17 \\ -6 & -8 \end{bmatrix}$$

We can establish a relationship between

A and D and

B and C.

Using the matrix

$$P = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}$$

But first, we will compute P^{-1}

$$P^{-1} = \frac{1}{\underset{ad-bc}{3-4}} \begin{bmatrix} 1 & -4 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}$$

Now, notice that

$$P^{-1}AP = \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 8 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = D$$

or $\boxed{P^{-1}AP = D}$

and

$$P^{-1}BP = \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 13 & 17 \\ -6 & -8 \end{bmatrix} = C$$

or $\boxed{P^{-1}BP = C}$

<u>Definition</u> $A_{n \times n}$, $B_{n \times n}$

A is Similar to B if there is an invertible $P_{n \times n}$ Such that

$$P^{-1}AP = B \qquad (3.1)$$

and we denote this relationship as

$$A \sim B.$$

An easy alternative to say that $A \sim B$ is to write (3.1) in its equivalence form:

$$AP = PB$$ (How is this equation obtained?)

<u>Theorem 4.21</u> $A_{n \times n}$, $B_{n \times n}$ and $C_{n \times n}$

i) $A \sim A$, ii) $A \sim B \Rightarrow B \sim A$.

iii) $A \sim B$ and $B \sim C \Rightarrow A \sim C$.

<u>Proof</u>. (Easy).

Thm 4.2.2

1) $A_{n\times n}$ and $B_{n\times n}$

2) $A \sim B$.

then,

i) $\det A = \det B$

ii) A invertible if and only if B is invertible

iii) A and B have the same rank.

iv) A and B have the same characteristic polyn.

v) A and B have the same eigenvalues.

vi) $A^m \sim B^m$, for all integers $m \geq 0$.

vii) If A is invertible, then $A^m \sim B^m$ for all integers m.

Proof.-

i) $\det B = \det(P^{-1}AP) = \det(P^{-1}) \det A \det P$

$$= \frac{1}{\cancel{\det P}} \det A \cancel{\det P} = \det A.$$

ii) Since $\det A = \det B$

then $\det A \neq 0 \Leftrightarrow \det B \neq 0$

$$\begin{array}{ccc} \det A \neq 0 & \Leftrightarrow & \det B \neq 0 \\ \Downarrow & & \Downarrow \\ A \text{ inv.} & \Longleftrightarrow & B \text{ inv} \end{array}$$

iii) $\det(B-\lambda I) = \det(P^{-1}AP - \lambda I) = \det(P^{-1}AP - \lambda P^{-1}IP)$

$$= \det(P^{-1}(A-\lambda I)P) \overset{(i)}{=} \det(A-\lambda I)$$

$B \sim C$ and $A \sim D$.

There is an important difference between C and D

THIS IS D is <u>diagonal</u>.

Why is that an advantage?

Because A is Similar to a diagonal matrix D, it receives a special name. This is the topic of the next definitions and theorems.

5.3 Diagonalization (*)

Consider
$$A=\begin{bmatrix}0 & -1 & -1\\ 1 & 2 & 1\\ -1 & -1 & 0\end{bmatrix}$$

and
$$P=\begin{bmatrix}1 & -1 & -1\\ -1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix},\quad P^{-1}=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 1\\ -1 & -1 & 0\end{bmatrix}$$

Compute the product: $P^{-1}AP$.

$$P^{-1}AP=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 1\\ -1 & -1 & 0\end{bmatrix}\begin{bmatrix}0 & -1 & -1\\ 1 & 2 & 1\\ -1 & -1 & 0\end{bmatrix}\begin{bmatrix}1 & -1 & -1\\ -1 & 1 & 0\\ 1 & 0 & 1\end{bmatrix}=$$

$$=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & 1\\ -1 & -1 & 0\end{bmatrix}\begin{bmatrix}0 & -1 & -1\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=D \text{ (diagonal matrix)}$$

It can be shown (DO It!) that

$\lambda_1=0$ is an eigenvalue with corresp. eigenvector $\vec{u}=\begin{bmatrix}1\\ -1\\ 1\end{bmatrix}$

$\lambda_2=1$ " " " " " eigenvectors $\vec{v}_1=\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix}$ and $\vec{v}_2=\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$

(*) Remark: In this section, we will consider matrices with real eigenvalues only. However, they may be repeated.

This implies

$$P = [\vec{u}\ \vec{v}_1\ \vec{v}_2] = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

and

$$D = P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

<u>Def.</u>

A matrix $A_{n\times n}$ such that there exists P invertible matrix satisfying

$$P^{-1}AP = D \text{ (diagonal matrix)} \qquad (2.1)$$

is said to be diagonalizable.

From the previous example we arrive to two important conclusion

1) A good candidate for an invertible matrix P satisfying (2.1) is a matrix whose columns are n linearly independent eigenvectors.

2) The diagonal matrix D has the eigenvalues along the diagonal.

Computing Eigenvalues and Eigenvectors of

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$|A - \lambda I| = 0 \iff \begin{vmatrix} -\lambda & -1 & -1 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0$$

$$\iff 0 = -\lambda \begin{vmatrix} 2-\lambda & 1 \\ -1 & -\lambda \end{vmatrix} + (-1)(-1)^3 \begin{vmatrix} 1 & 1 \\ -1 & -\lambda \end{vmatrix} + (-1) \begin{vmatrix} 1 & 2-\lambda \\ -1 & -1 \end{vmatrix}$$

$$= -\lambda\left(\lambda(\lambda-2)+1\right) + (-\lambda+1) - (-1+2-\lambda) =$$

$$= -\lambda^2(\lambda-2) \overset{-\lambda}{-} \cancel{\lambda} + \cancel{1} + 1 - \cancel{2} + \cancel{\lambda} =$$

$$= \lambda\left(-\lambda(\lambda-2) - 1\right) = \lambda\left(-\lambda^2 + 2\lambda - 1\right)$$

$$= -\lambda\left(\lambda^2 - 2\lambda + 1\right) = -\lambda(\lambda-1)^2$$

or $\boxed{\lambda(\lambda-1)^2 = 0}$ characteristic equation.

$\lambda_1 = 0$, $\lambda_2 = 1$ (algebraic multiplicity 2)

Finding eigenvectors:

a) $\lambda_1 = 0$

$$(A - \overset{=0}{\lambda} I)\vec{x} = \vec{0} \iff A\vec{x} = \vec{0}.$$

or augmented matrix:

$$\left[\begin{array}{ccc|c} 0 & -1 & -1 & 0 \\ 1 & 2 & 1 & 0 \\ -1 & -1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \boxed{x_2 = -x_3}$$

$$x_1 = -2x_2 - x_3$$

$$x_1 = 2x_3 - x_3 = x_3$$

or $\boxed{x_1 = x_3}$

Then eigenvectors for $\lambda_1 = 0$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \checkmark$$

Basis for Eigenspace of $\lambda_1 = 0 = \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$

b) $\underline{\lambda_2 = 1}$ (algebraic multiplicity 2)

$$(A - \overset{=1}{\lambda_2} I)\vec{x} = \vec{0} \Leftrightarrow \boxed{(A - I)\vec{x} = \vec{0}}$$

Solve this Linear syst.

or augmented matrix

$$\left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Free Variables

$$x_1 = -x_2 - x_3 \Rightarrow$$

Eigenvectors are:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

then

$$E_{\lambda_2=1} = \left\{ \vec{x} \in \mathbb{R}^3 : \vec{x} = t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \ t, s \in \mathbb{R} \right\}$$

This implies geometric multiplicity 2.

Thm 5 Diagonalization Thm.

A matrix $A_{n\times n}$ is diagonalizable. It means there exists P invertible such that

$$\boxed{P^{-1}AP = D} \text{ (diagonal matrix)}$$

if and only if A has n linearly independent eigenvectors.

It will be shown that Columns of P are n linearly independent eigenvectors of A and $D_{n\times n}$ is a diagonal matrix with the Corresponding eigenvalues along the diagonal.

Before proving this theorem observe the following:

If $P = [\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_n]$, $A_{n\times n}$, and

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ & & \ddots & 0 \\ & 0 & & \lambda_n \end{bmatrix}$$
(diagonal Matrix)

Then,

$$AP = A[\vec{v}_1 \dots \vec{v}_n] = [A\vec{v}_1 \ A\vec{v}_2 \dots A\vec{v}_n] \qquad (3.1)$$

and

$$PD = [\vec{v}_1 \cdots \vec{v}_n]D = P\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = [\lambda_1\vec{v}_1 \ \lambda_2\vec{v}_2 \dots \lambda_n\vec{v}_n] \qquad (3.2)$$

Proof thm 5.-

$$A_{n\times n} \text{ diagonalizable } \overset{\text{definition}}{\iff} \text{There exists } P = [\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_n]$$
invertible such that $P^{-1}AP = D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \iff$
There exists P invertible such that $AP = PD$

$$\overset{(3.1),(3.2)}{\iff} \quad [A\vec{v}_1\ A\vec{v}_2 \cdots A\vec{v}_n] = [\lambda_1\vec{v}_1\ \lambda_2\vec{v}_2 \cdots \lambda_n\vec{v}_n]$$
and $\{\vec{v}_1, \vec{v}_2, .., \vec{v}_n\}$ is linearly independent.

$\iff$ $A\vec{v}_1 = \lambda_1\vec{v}_1,\ A\vec{v}_2 = \lambda_2\vec{v}_2, \ldots,\ A\vec{v}_n = \lambda_n\vec{v}_n$
and $\{\vec{v}_1, \vec{v}_2 .., \vec{v}_n\}$ is lin. independent

$\iff$ A has n linearly independent eigenvectors
$\vec{v}_1, \vec{v}_2, .., \vec{v}_n$, with corresponding eigenvalues
$\lambda_1, \lambda_2, , \lambda_n$ (possibly repeated).

Thm 5 suggests a <u>method to diagonalize a matrix A</u> if possible. (Review previous example).

1) Find eigenvalues of A: $\lambda_1, \lambda_2, .., \lambda_n$ (possibly repeated)
$\{\vec{v}_1, \vec{v}_2, .., \vec{v}_n\}$

2) Find n linearly independent eigenvectors of A, if possible. Otherwise, A is not diagonalizable according to thm 5.

3) Construct P from eigenvectors obtained in step 2.

More precisely, define

$$P = [\vec{v}_1 \ \vec{v}_2 \cdots \vec{v}_n]$$

Clearly, P is invertible because the set of eigenvectors $S = \{\vec{v}_1 .., \vec{v}_n\}$ is lin. indep.

and also

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & \ddots & 0 \\ 0 & & \lambda_n \end{bmatrix}$$

where $\lambda_1 .., \lambda_n$ are the eigenvalues of A corresponding to $\vec{v}_1 .., \vec{v}_n$ (possibly repeated).

Thm 2 (Section 5.1)

If $A_{n\times n}$ matrix is such that

eigenvectors

$$A\vec{v}_1 = \lambda_1 \vec{v}_1, .., \quad A\vec{v}_r = \lambda_r \vec{v}_r, \quad \vec{v}_i \neq \vec{0}, \ i=1,2..,r$$

and $\lambda_1, .., \lambda_r$ are distinct eigenvalues,

then the set $\{\vec{v}_1, .., \vec{v}_r\}$ is linearly independent.

Proof.-

Want to prove

If $c_1\vec{v}_1 + c_2\vec{v}_2 + .. + c_r\vec{v}_r = \vec{0} \Rightarrow c_1 = 0, .., c_r = 0.$

So, start assuming

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_r\vec{v}_r = \vec{0} \qquad (6.1)$$

and multiply by A both sides

$$A(c_1\vec{v}_1 + .. + c_r\vec{v}_r) = A\vec{0} = \vec{0}$$

Using matrix operations and the hypothesis $\underset{\text{eigenval.}}{\lambda\text{'s}}$ and $\underset{\text{eigenvect.}}{\vec{v}\text{'s}}$

$$c_1 A\vec{v}_1 + .. + c_r A\vec{v}_r = \vec{0} \Leftrightarrow c_1\lambda_1\vec{v}_1 + .. + c_r\lambda_r\vec{v}_r = \vec{0}$$

multiplying $\boxed{\text{Equ. } (6.1) \times \lambda_1}$ and subtracting $-c_1\lambda_1\vec{v}_1 - \cdots - c_r\lambda_1\vec{v}_r = 0$

$$\boxed{c_2(\lambda_2 - \lambda_1)\vec{v}_2 + .. + c_r(\lambda_r - \lambda_1)\vec{v}_r = \vec{0}}$$

$\overset{*}{\Rightarrow} \boxed{c_2 = 0, .., c_r = 0}$, Since $\lambda_2 \neq \lambda_1, .., \lambda_r \neq \lambda_1$ and $\vec{v}_2, .., \vec{v}_r \neq \vec{0}$.

Substitution into (6.1) $\Rightarrow \boxed{c_1 = 0.}$

* This conclusion requires an induction hypothesis. NOT COMPLETE! But, very easy to do!

Theorem 6

Any $A_{n\times n}$ matrix with n distinct eigenvalues: $\lambda_1, .., \lambda_n$ is diagonalizable.

Proof.- Trivial consequence of thm 2 and thm 5.

If A has n distinct eigenvalues $\overset{\text{Thm 2}}{\Longrightarrow}$ The set $S = \{\vec{v}_1, .., \vec{v}_n\}$ of the eigenvectors corresponding to the n distinct eigenvalues is linearly independent.

$\overset{\text{Thm 5}}{\Longrightarrow}$ A is diagonalizable.

Why the reciprocal statement:

$$A \text{ diagonalizable} \Longrightarrow A \text{ has } n \text{ distinct eigenvalues}$$

is not TRUE? See our previous example!

Thm 4.24 1) $A_{n\times n}$ and $\lambda_1, .., \lambda_k$ distinct eigenvalues of A.

2) B_i is a basis for Eigenspace λ_i, $i=1,..,k$

Then,

$$B = B_1 \cup B_2 \cup .. \cup B_k$$

is a linearly independent set of vectors.

Remark: B is a set formed by the basis vectors for all of the eigenspaces.

Consider

Ex #13 (5.1) $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

Determine if A is diagonalizable or not.

1) First compute eigenvalues of A

$$|A-\lambda I| = 0 \iff \begin{vmatrix} 4-\lambda & 0 & 1 \\ -2 & 1-\lambda & 0 \\ -2 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\iff (4-\lambda)(1-\lambda)(1-\lambda) - [-2(1-\lambda)] = 0$$

$$\iff (1-\lambda)[(4-\lambda)(1-\lambda)+2] = (1-\lambda)[\lambda^2 - 5\lambda + \underbrace{4+2}_{6}]$$

$$= (1-\lambda)[(\lambda-3)(\lambda-2)] = 0$$

$$\iff \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$$

Since $A_{3\times 3}$ has 3 distinct eigenvalues by thm 6

<u>A is diagonalizable.</u> ✓

If we also would like to find a matrix P such that

$$P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

We need to find the eigenvectors associated to each eigenvalue

For $\lambda_1 = 1$ $\quad \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$,

For $\lambda_2 = 2$ $\quad \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

For $\lambda_3 = 3$ $\quad \vec{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

Then

$$P = \begin{array}{c} \begin{matrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{matrix} \\ \underbrace{\begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}}_{\text{order of the columns is important}} \end{array} ,$$

Why we don't need to verify that the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent?

Also,

$$P^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -2 & 0 & -1 \end{bmatrix}$$

$$P^{-1}A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 2 \\ -6 & 0 & -3 \end{bmatrix}$$

$$(P^{-1}A)P = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 0 & 2 \\ -6 & 0 & -3 \end{bmatrix} \begin{bmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$